doi:10.1006/jmaa.2001.7733, available online at http://www.idealibrary.com on IDEAL®

Further Results on the Stability of Linear Systems with Multiple Delays[1]

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Submitted by C. T. Leondes

Received February 21, 2001; published online January 24, 2002

This paper has two purposes. One is to establish new stability criteria for linear systems with multiple time-varying/constant delays, and the other is to point out the limitations of the author's previous results. The stability criteria, including delay-dependent and delay-independent ones, are derived by using both time-domain and frequency-domain techniques. All of the established stability criteria are given in the form of linear matrix inequality problems, including only one tuning parameter matrix $P > 0 \in R^{n \times n}$. © 2002 Elsevier Science (USA)

1. INTRODUCTION

Recently, new stability theorems for retarded dynamical systems have been established in [1]. A new analysis technique has also been proposed in [1] for estimating the derivative of Lyapunov function along the solution of a system at some specific instant. As the applications of these new stability theorems, both delay-dependent and delay-independent stability conditions

[1] This work was supported by National Science Foundation of China Project 60074026 and Guangdong Province Natural Science Foundation of China Project 000409.

0022-247X/02 $35.00

for a following linear system with multiple time-varying delays,

$$\dot{x}(t) = Ax(t) + \sum_{k=1}^{m} A_k x(t - \tau_k(t)), \qquad t \geq t_0 \in R$$
$$x_{t_0}(\theta) = x(t_0 + \theta) = \phi(\theta), \tag{1}$$
$$t_0 + \theta \in E_{t_0} = \bigcup_{k=1}^{m} \{t - \tau_k(t) \mid t - \tau_k(t) \leq t_0, t \geq t_0\} \cup \{t_0\},$$

have been presented in [1], where $\phi \in C_n$, $x \in R^n$, $A \in R^{n\times n}$ and $A_k \in R^{n\times n}$, and $\tau_k(t) \leq \tau_{kM} < \infty$ with constant $\tau_{kM} > 0$ for $k = 1, 2, \ldots, m$ are the time-varying and bounded delays. For the special case of system (1),

$$\dot{x}(t) = Ax(t) + \sum_{k=1}^{m} A_k x(t - \tau_k), \qquad t \geq 0, \tag{2}$$

with unknown but constant delays $\tau_k \leq \tau_{kM} < \infty$ for $k = 1, 2, \ldots, m$, the same stability conditions have also been established in [2] by use of frequency-domain techniques. The main purposes of this paper are to establish new stability conditions for systems (1) and (2) and, at the same time, to point out that the limitations of the previous stability conditions established in [1] and [2], respectively. All of the stability definitions and the notations used in this paper are the same ones as those defined in [1] and [2], respectively.

2. STABILITY CRITERIA

In the following, the stability criteria for systems (1) and (2) are derived by using time-domain and frequency-domain techniques, respectively. All of the stability criteria are given in the form of linear matrix inequality (LMI) problems, including only one tuning parameter matrix, $P > 0 \in R^{n\times n}$.

First, we give the following lemmas, and, for the proofs of Lemma 1 and Lemma 2, see [1] and [3], respectively.

LEMMA 1. *Let $P > 0 \in R^{n\times n}$ be a constant matrix; let $T_k \in R^{n\times n}$ for $k = 0, 1, \ldots, m$ be symmetric constant matrices; and let $\alpha \in R_+$ and $\eta_{kM} > 0$ for $k = 1, 2, \ldots, m$ be real numbers. Then,*

$$T_0 + \sum_{k=1}^{m} \eta_k T_k \leq -\alpha P \tag{3}$$

holds for all $\eta_k \in [-\eta_{kM}, \eta_{kM}]$, $k = 1, 2, \ldots, m$, if and only if inequality (3) *holds at all of the 2^m vertices of the following hyper-rectangle*:

$$H := \big\{\eta = (\eta_1 \eta_2 \ldots \eta_m)^T \in R^m \mid \eta_k \in [-\eta_{kM}, \eta_{kM}], \; k = 1, 2, \ldots, m\big\}. \tag{4}$$

LEMMA 2. *Let* $A, B \in R^{n\times n}$ *and* $\alpha \in R_+$. *Then, the eigenvalues of* $A + jB$ *are in the region of* $\operatorname{Re} s \le -\alpha$ *in the complex plane if and only if the eigenvalues of the* $2n \times 2n$ *real matrix*

$$\begin{bmatrix} A & B \\ -B & A \end{bmatrix} \in R^{2n\times 2n} \tag{5}$$

are in the same region.

LEMMA 3. *Let* $A = A^T \in R^{n\times n}$ *and* $B = -B^T \in R^{n\times n}$ *so that* $U = A + jB = U^* \in C^{n\times n}$. *Then,* $v^*(A + jB)v \le 0$ (*or* < 0) *for any* $v \in C^n$ *if and only if*

$$W = \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \le 0 \qquad (\text{or} < 0). \tag{6}$$

Proof. Let $\lambda_{\max}(U)y = Uy$ and $\lambda_{\max}(W)z = Wz$, where $y \in C^n$, $z \in C^{2n}$, and $\lambda_{\max}(U) \in R$ and $\lambda_{\max}(W) \in R$ denote the maximum eigenvalues of U and W, respectively. Then by Lemma 2, it is easy to see that

$$\lambda_{\max}(U) = \frac{y^*Uy}{y^*y} = \lambda_{\max}(W) = \frac{z^*Wz}{z^*z} = \max_{v^*v=1}\left\{\frac{v^*Uv}{v^*v}\right\} = \max_{u^*u=1}\left\{\frac{u^*Wu}{u^*u}\right\}. \tag{7}$$

The proof is completed. Q.E.D.

THEOREM 1. *Let* $\gamma > 0$ *and* $\tau_{kM} > 0$ *for* $k = 1, 2, \ldots, m$. *The equilibrium* $x^* \equiv 0$ *of system* (1) *is exponentially stable with respect to the constant decay degree* $\gamma/2 > 0$ *if there is a positive definite matrix* $P > 0 \in R^{n\times n}$ *such that*

$$\begin{bmatrix} -PA - A^TP - \gamma P - \sum_{k=1}^{m} e^{\frac{1}{2}\gamma\tau_{kM}}P & e^{\frac{1}{2}\gamma\tau_{1M}}PA_1 & \cdots & e^{\frac{1}{2}\gamma\tau_{mM}}PA_m \\ e^{\frac{1}{2}\gamma\tau_{1M}}A_1^TP & e^{\frac{1}{2}\gamma\tau_{1M}}P & & 0 \\ \vdots & & \ddots & \\ e^{\frac{1}{2}\gamma\tau_{mM}}A_m^TP & 0 & & e^{\frac{1}{2}\gamma\tau_{mM}}P \end{bmatrix} \ge 0. \tag{8}$$

Proof. Let $V(x) = x^TPx$. Then,

$$\dot V(x_t(0)) = x_t^T(0)(PA + A^TP)x_t(0) + \sum_{k=1}^{m} 2x_t^T(0)PA_kx_t(-\tau_k(t)). \tag{9}$$

Note that by using standard Schur complement arguments [4], condition (8) implies

$$PA + A^TP + \sum_{k=1}^{m} e^{\frac{1}{2}\gamma\tau_{kM}}(PA_kP^{-1}A_k^TP + P) \le -\gamma P. \tag{10}$$

Note that $2ab \le a^2 + b^2$ for any $a, b \in R$. By Cauchy–Schwarz inequality, Eqs. (9) and (10), it is easy to see that condition (8) implies

$$
\begin{aligned}
\dot V(y_t(0)) &= y_t^T(0)(PA + A^T P)y_t(0) + \sum_{k=1}^{m} 2y_t^T(0)PA_k y_t(-\tau_k(t)) \\
&\le y_t^T(0)(PA + A^T P)y_t(0) \\
&\quad + \sum_{k=1}^{m} 2\big(y_t^T(0)PA_k P^{-1} A_k^T P y_t(0) y_t^T(-\tau_k(t))P y_t(-\tau_k(t))\big)^{1/2} \\
&\le y_t^T(0)(PA + A^T P)y_t(0) \\
&\quad + \sum_{k=1}^{m} 2e^{\frac{1}{2}\gamma\tau_k(t)}\big(y_t^T(0)PA_k P^{-1} A_k^T P y_t(0) y_t^T(0)P y_t(0)\big)^{1/2} \\
&\le y_t^T(0)(PA + A^T P)y_t(0) \\
&\quad + \sum_{k=1}^{m} e^{\frac{1}{2}\gamma\tau_{kM}} y_t^T(0)(PA_k P^{-1} A_k^T P + P)y_t(0) \\
&\le -\gamma V(y_t(0)), \qquad \forall\, y_t \in S(L_t(\theta)),
\end{aligned} \tag{11}
$$

whenever $V(x_t(0)) = \overline{V}_{t_0} \exp\{-\gamma(t - t_0)\}$ on $t \ge t_0 \in R$, where

$$
\|P^{1/2} y_t(-\tau_k(t))\|_2^2 = \left\| P^{1/2} y_t(0) \exp\left\{\frac{1}{2}\gamma(\tau_k(t) + \theta_{Xk})\right\}\right\|_2^2, \tag{12}
$$

with $\theta_{Xk} \in (-\infty, 0]$ for all $k = 1, 2, \ldots, m$ and $S(L_t(\theta))$ is defined in [1] as follows:

$$
S(L_t(\theta)) = \left\{ y_t \in C_n \;\middle|\; \begin{array}{r} L_t(\theta) = L(t+\theta) = \overline{V}_{t_0} \exp\{-\gamma(t+\theta-t_0)\} \\ V(y_t(0)) = L_t(0) \\ \|P^{1/2} y_t(\theta)\|_2^2 = \left\| P^{1/2} y_t(0) \exp\left\{-\frac{1}{2}\gamma(\theta - \theta_X)\right\}\right\|_2^2 \\ \theta \in [-\tau, 0], \quad \theta_X \in (-\infty, 0] \end{array} \right\}. \tag{13}
$$

By Theorem 3.3 in [1], the proof is completed. Q.E.D.

Remark 1. In the previous paper [1], the stability condition corresponding to (8) is

$$
PA + A^T P + \left[\sum_{k=1}^{m} \pm e^{\frac{1}{2}\gamma\tau_{kM}}(PA_k + A_k^T P)\right]_l \le -\gamma P, \quad \forall\, l = 1, 2, \ldots, 2^m, \tag{14}
$$

where $[\sum_{k=1}^{m} \pm e^{\frac{1}{2}\gamma\tau_{kM}}(PA_k + A_k^T P)]_l$ for $l = 1, 2, \ldots, 2^m$ denotes all 2^m cases of alternating sign. Based on the proof of Theorem 1, it can be seen that condition (14) guarantees the exponential stability of system (1) with respect to the constant decay degree $\gamma/2 > 0$ for the following cases of system (1):

(i) The dimension of system (1) is one, i.e., $n = 1$.

(ii) $\tau_{kM} > 0$ for $k = 1, 2, \ldots, m$ is sufficiently small that

$$\sum_{k=1}^{m} 2x_t^T(0)PA_k x_t(-\tau_k(t)) \leq \sum_{k=1}^{m} 2e^{\frac{1}{2}\gamma\tau_{kM}} \left|x_t^T(0)PA_k x_t(0)\right|, \quad \tau_k(t) \in [0, \tau_{kM}], \tag{15}$$

whenever $V(x_t(0)) = \overline{V}_{t_0} \exp\{-\gamma(t - t_0)\}$ on $t \geq t_0 \in R$. However, determining exactly $\tau_{kM} > 0$ for $k = 1, 2, \ldots, m$ such that (14) together with (15) holds is still an open problem.

THEOREM 2. *The equilibrium* $x^* \equiv 0$ *of system* (1) *is globally uniformly asymptotically stable if there is a positive definite matrix* $P > 0 \in R^{n\times n}$ *such that*

$$\begin{bmatrix} -PA - A^T P - mP & PA_1 & \cdots & PA_m \\ A_1^T P & P & & 0 \\ \vdots & & \ddots & \\ A_m^T P & 0 & & P \end{bmatrix} > 0 \tag{16}$$

and globally uniformly stable if

$$\begin{bmatrix} -PA - A^T P - mP & PA_1 & \cdots & PA_m \\ A_1^T P & P & & 0 \\ \vdots & & \ddots & \\ A_m^T P & 0 & & P \end{bmatrix} \geq 0. \tag{17}$$

Proof. Note that for any positive $\tau_{kM} > 0$, $k = 1, 2, \ldots, m$, there is a sufficiently small number $\varepsilon > 0$ such that condition (16) implies

$$PA + A^T P + \sum_{k=1}^{m} e^{\frac{1}{2}\varepsilon\tau_{kM}}(PA_k P^{-1} A_k^T P + P) \leq -\varepsilon P, \tag{18}$$

which is equivalent to

$$\begin{bmatrix} -PA - A^T P - \varepsilon P - \sum_{k=1}^{m} e^{\frac{1}{2}\varepsilon\tau_{kM}} P & e^{\frac{1}{2}\varepsilon\tau_{1M}} PA_1 & \cdots & e^{\frac{1}{2}\varepsilon\tau_{mM}} PA_m \\ e^{\frac{1}{2}\varepsilon\tau_{1M}} A_1^T P & e^{\frac{1}{2}\varepsilon\tau_{1M}} P & & 0 \\ \vdots & & \ddots & \\ e^{\frac{1}{2}\varepsilon\tau_{mM}} A_m^T P & 0 & & e^{\frac{1}{2}\varepsilon\tau_{mM}} P \end{bmatrix} > 0. \tag{19}$$

By Theorem 1 here and Theorem 3.4 in [1], the results are proved almost immediately. Q.E.D.

It should be noted that Theorem 1 and Theorem 2 are also suitable to system (2). To compare the results established in [2], let us establish the corresponding stability criteria by the frequency-domain approach.

THEOREM 3. *Let* $\alpha = \gamma/2 > 0$ *with* $\gamma > 0$ *and* $\tau_{kM} > 0$ *for* $k = 1, 2, \ldots, m$. *System* (2) *is* α*-stable locally in the delays if there is a constant matrix* $P > 0 \in R^{n\times n}$ *such that condition* (8) *holds.*

Proof. Let $\tau_k \in [0, \tau_{kM}]$ for $k = 1, 2, \ldots, m$, $s = \sigma + j\omega \in C$ with $\sigma \in R$ and $\omega \in R$, where

$$\begin{aligned} W(\tau_k, s) &= A + \sum_{k=1}^{m} e^{-\tau_k s} A_k \\ &= A + \sum_{k=1}^{m} \cos(-\tau_k\omega) e^{-\tau_k\sigma} A_k + j\sum_{k=1}^{m} \sin(-\tau_k\omega) e^{-\tau_k\sigma} A_k, \end{aligned} \tag{20}$$

and $\tilde{y} := \tilde{y}(\tau_k, \tilde{s}) \in C^n$ satisfy $W(\tau_k, \tilde{s})\tilde{y} = \lambda(W(\tau_k, \tilde{s}))\tilde{y}$ at $s = \tilde{s} = \tilde{\sigma} + j\tilde{\omega} \in C$. By (20) and the Cauchy–Schwarz inequality, one obtains

$$\begin{aligned} \tilde{\sigma} = \operatorname{Re}\lambda(W(\tau_k, \tilde{s})) &= \frac{\tilde{y}^*(\widetilde{P}W(\tau_k, \tilde{s}) + W^*(\tau_k, \tilde{s})\widetilde{P})\tilde{y}}{2\tilde{y}^*\widetilde{P}\tilde{y}} \\ &= \frac{\tilde{y}^*(\widetilde{P}A + A^T\widetilde{P})\tilde{y} + \sum_{k=1}^{m} e^{-\tau_k\tilde{\sigma}}\tilde{y}^*(\widetilde{P}A_k e^{-j\tau_k\tilde{\omega}} + A_k^T\widetilde{P}e^{j\tau_k\tilde{\omega}})\tilde{y}}{2\tilde{y}^*\widetilde{P}\tilde{y}} \\ &\le \frac{\tilde{y}^*(\widetilde{P}A + A^T\widetilde{P})\tilde{y} + \sum_{k=1}^{m} e^{-\tau_k\tilde{\sigma}}\big(|\tilde{y}^*\widetilde{P}A_k\tilde{y}| + |\tilde{y}^*A_k^T\widetilde{P}\tilde{y}|\big)}{2\tilde{y}^*\widetilde{P}\tilde{y}} \\ &\le \frac{\tilde{y}^*(\widetilde{P}A + A^T\widetilde{P})\tilde{y} + \sum_{k=1}^{m} 2e^{-\tau_k\tilde{\sigma}}\big(\tilde{y}^*\widetilde{P}A_k\widetilde{P}^{-1}A_k^T\widetilde{P}\tilde{y}\tilde{y}^*\widetilde{P}\tilde{y}\big)^{1/2}}{2\tilde{y}^*\widetilde{P}\tilde{y}} \\ &\le \frac{\tilde{y}^*(\widetilde{P}A + A^T\widetilde{P})\tilde{y} + \sum_{k=1}^{m} e^{-\tau_k\tilde{\sigma}}\tilde{y}^*\big(\widetilde{P}A_k\widetilde{P}^{-1}A_k^T\widetilde{P} + \widetilde{P}\big)\tilde{y}}{2\tilde{y}^*\widetilde{P}\tilde{y}} \end{aligned} \tag{21}$$

for any $\widetilde{P} > 0 \in R^{n\times n}$. Let us prove the theorem by contradiction. Assume that condition (8) holds. If there is a root $\tilde{s} = \tilde{\sigma} + j\tilde{\omega} \in C$ of the corresponding characteristic equation

$$\det\left[sI_n - \left(A + \sum_{k=1}^{m} e^{-\tau_k s} A_k\right)\right] = 0 \tag{22}$$

for $\tau_k \in [0, \tau_{kM}]$, $k = 1, 2, \ldots, m$, such that

$$\tilde{\sigma} > -\alpha, \tag{23}$$

then by condition (8) and Eq. (21), one has

$$-\alpha < \tilde{\sigma} \leq \frac{\tilde{y}^*(PA + A^T P)\tilde{y} + \sum_{k=1}^{m} e^{-\tau_k \tilde{\sigma}} \tilde{y}^*(PA_k P^{-1} A_k^T P + P)\tilde{y}}{2\tilde{y}^* P\tilde{y}}$$

$$\leq \frac{\tilde{y}^*(PA + A^T P)\tilde{y} + \sum_{k=1}^{m} e^{\alpha\tau_{kM}} \tilde{y}^*(PA_k P^{-1} A_k^T P + P)\tilde{y}}{2\tilde{y}^* P\tilde{y}} \leq -\alpha, \quad (24)$$

which is a contradiction. The proof is completed. Q.E.D.

THEOREM 4. *System* (2) *is* ε*-stable independently of the delays if there is a positive definite matrix* $P > 0 \in R^{n\times n}$ *such that condition* (16) *holds*.

Proof. The proof is almost immediate from the proofs of Theorem 3 and Theorem 2. Q.E.D.

Remark 2. In the previous paper [2], the corresponding stability condition for system (2) is given as

$$PA + A^T P + \left[\sum_{k=1}^{m} \pm e^{\alpha\tau_{kM}} (PA_k + A_k^T P)\right]_l \leq -2\alpha P, \qquad \forall l = 1, 2, \ldots, 2^m, \quad (25)$$

where $[\sum_{k=1}^{m} \pm e^{\alpha\tau_{kM}} (PA_k + A_k^T P)]_l$ for $l = 1, 2, \ldots, 2^m$ denotes all 2^m cases of alternating sign. Based on the proof of Theorem 3, it can be seen that condition (25) guarantees the α-stability of system (2) with any $\tau_k \in [0, \tau_{kM}]$ for the following cases:

(i) The dimension of system (2) is one, i.e., $n = 1$.

(ii) $\tau_{kM} > 0$ for $k = 1, 2, \ldots, m$ is sufficiently small that

$$\sum_{k=1}^{m} \frac{e^{-\tau_k \tilde{s}} \tilde{y}^* PA_k \tilde{y}}{\tilde{y}^* P\tilde{y}} \leq \sum_{k=1}^{m} \frac{|e^{-\tau_k \tilde{s}} \tilde{y}^* PA_k \tilde{y}|}{\tilde{y}^* P\tilde{y}}, \qquad \forall\, \tau_k \in [0, \tau_{kM}]. \quad (26)$$

To solve the problem in case (ii) of Remark 2, we give the following result.

THEOREM 5. *Let* $\alpha > 0$ *and* $\tau_{kM} > 0$ *for* $k = 1, 2, \ldots, m$. *System* (2) *is* α*-stable locally in the delays if there is a constant matrix* $P > 0 \in R^{n\times n}$ *such that*

$$\begin{bmatrix} T_0 + 2\alpha P & 0 \\ 0 & T_0 + 2\alpha P \end{bmatrix} + \left\{ \sum_{k=1}^{m} \pm e^{\alpha\tau_{kM}} \begin{bmatrix} T_k & 0 \\ 0 & T_k \end{bmatrix} \right.$$
$$\left. + \sum_{k=1}^{m} \pm e^{\alpha\tau_{kM}} \begin{bmatrix} 0 & -U_k \\ U_k & 0 \end{bmatrix} \right\}_l \leq 0 \quad (27)$$

for all cases of $l \in \{1, 2, \ldots, 2^{2m}\}$, *where* $T_0 = PA + A^T P$, $T_k = PA_k + A_k^T P$, $U_k = PA_k - A_k^T P$ *for* $k = 1, 2, \ldots, m$, *and*

$$\left\{ \sum_{k=1}^{m} \pm e^{\alpha\tau_{kM}} \begin{bmatrix} T_k & 0 \\ 0 & T_k \end{bmatrix} + \sum_{k=1}^{m} \pm e^{\alpha\tau_{kM}} \begin{bmatrix} 0 & -U_k \\ U_k & 0 \end{bmatrix} \right\}_l \quad (28)$$

for $l = 1, 2, \ldots, 2^{2m}$ *denotes all* 2^{2m} *cases of alternating sign.*

Proof. Let us start the proof from Eqs. (20) and (21) with $\widetilde{P} = P$ in the proof of Theorem 3. Then, we have

$$\tilde{\sigma} = \mathrm{Re}\lambda(W(\tau_k, \tilde{s})) = \frac{\tilde{y}^*(PW(\tau_k, \tilde{s}) + W^*(\tau_k, \tilde{s})P)\tilde{y}}{2\tilde{y}^*\widetilde{P}\tilde{y}}$$
$$= \frac{\tilde{y}^*[T_0 + \sum_{k=1}^m e^{-\tau_k\tilde{\sigma}}\cos(\tau_k\tilde{\omega})T_k - j\sum_{k=1}^m e^{-\tau_k\tilde{\sigma}}\sin(\tau_k\tilde{\omega})U_k]\tilde{y}}{2\tilde{y}^*P\tilde{y}}. \quad (29)$$

By Lemma 1, Lemma 2, and Lemma 3 together with (29), it is not difficult to see that condition (27) implies

$$\max_{v\in C^n}\left\{\frac{v^*[T_0 + e^{\alpha\tau_{kM}}(\sum_{k=1}^m\cos(\tau_k\tilde{\omega})T_k - j\sum_{k=1}^m\sin(\tau_k\tilde{\omega})U_k)]v}{2v^*Pv}\right\}$$
$$\le -\alpha, \qquad \forall\,\tau_k \in [0, \tau_{kM}], \quad k = 1, 2, \dots, m. \quad (30)$$

Assume that condition (27) holds. If there is a root $\tilde{s} = \tilde{\sigma} + j\tilde{\omega} \in C$ of the corresponding characteristic equation (22) for $\tau_k \in [0, \tau_{kM}]$, $k = 1, 2, \dots, m$, such that

$$\tilde{\sigma} > -\alpha, \quad (31)$$

then by Lemma 1, Lemma 2, Lemma 3, and Eqs. (29) and (30), one has

$$-\alpha < \tilde{\sigma} \le \frac{\tilde{y}^*[T_0 + \sum_{k=1}^m e^{-\tau_k\tilde{\sigma}}\cos(\tau_k\tilde{\omega})T_k - j\sum_{k=1}^m e^{-\tau_k\tilde{\sigma}}\sin(\tau_k\tilde{\omega})U_k]\tilde{y}}{2\tilde{y}^*P\tilde{y}}$$
$$\le \max_{v\in C^n}\left\{\frac{v^*[T_0 + e^{\alpha\tau_{kM}}(\sum_{k=1}^m\cos(\tau_k\tilde{\omega})T_k - j\sum_{k=1}^m\sin(\tau_k\tilde{\omega})U_k)]v}{2v^*Pv}\right\}$$
$$\le -\alpha, \qquad \forall\tau_k \in [0, \tau_{kM}], \quad k = 1, 2, \dots, m, \quad (32)$$

which is a contradiction. The proof is completed. Q.E.D.

THEOREM 6. *System (2) is ε-stable independently of the delays if there is a constant matrix $P > 0 \in R^{n\times n}$ such that*

$$\begin{bmatrix} T_0 & 0 \\ 0 & T_0 \end{bmatrix} + \left\{\sum_{k=1}^m \pm\begin{bmatrix} T_k & 0 \\ 0 & T_k \end{bmatrix} + \sum_{k=1}^m \pm\begin{bmatrix} 0 & -U_k \\ U_k & 0 \end{bmatrix}\right\}_l < 0 \quad (33)$$

for all cases of $l \in \{1, 2, \dots, 2^{2m}\}$, where $T_0 = PA + A^TP$, $T_k = PA_k + A_k^TP$, $U_k = PA_k - A_k^TP$ for $k = 1, 2, \dots, m$, and

$$\left\{\sum_{k=1}^m \pm\begin{bmatrix} T_k & 0 \\ 0 & T_k \end{bmatrix} + \sum_{k=1}^m \pm\begin{bmatrix} 0 & -U_k \\ U_k & 0 \end{bmatrix}\right\}_l \quad (34)$$

for $l = 1, 2, \dots, 2^{2m}$ denotes all 2^{2m} cases of alternating sign.

Proof. The proof can be obtained similarly from the proof of Theorem 5. Q.E.D.

Remark 3. It should be noted that all of the stability criteria established above can be solved easily by using the corresponding programs in the MATLAB LMI Toolbox [4]. However, it should be solved in an iterative way if one wants to get the largest $\alpha = \gamma/2 > 0$ in (8) and (27) for given $\tau_{kM} > 0$ for $k = 1, 2, \ldots, m$.

3. CONCLUSION

New stability criteria, including both delay-dependent and delay-independent ones for linear systems with multiple time-varying/constant delays, have been established by the use of time-domain and frequency-domain techniques, respectively. In addition, the limitations of the author's results presented in previous papers [1, 2] have been pointed out. All of the established stability criteria have been presented in the form of LMI problems including only one tuning parameter matrix $P > 0 \in R^{n \times n}$. Therefore, they are very easy to use in various applications.

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